Maximal Compactness VIA Ideals

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Abstract: *In this article we introduce and characterize maximal compactness, maximal countably compact and maximal lindelof spaces in ideal topological spaces and show that Tyckonof Theorem is not true for maximal I-compact spaces.*

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I. Introduction and preliminaries

An ideal **[3,8]** on a topological space (X,T,I) is a nonempty collection of subsets of X , satisfying
(i) A∈ I and B⊂A implies B ∈ I (heredity).
(ii) A ∈ I and B ∈ I implies A ∪ B ∈ I (finite additivity).
If I is an ideal on a topological space(X,T,I) then (X,T,I) is called an idealtopological space**.** If (X,T,I) is an ideal topological space, then a collection $\{U_i : i \in K\}$ of subsets of X is called an I-cover iff it has a finite subcollection

$$\{U_i : i \in K_o\}, \text{where } K_o \text{ is a finite subset of } K.$$

such that $X \setminus \cup \{U_i : i \in K_o\} \in I$. (X,T,I) is called compact modulo I or I-compact **[1,5,6]** iff every open cover of X is an I-cover. An I-compact ideal topological space (X,T,I) is called maximal I-compact iff for every topology U on X strictly finer than T the ideal topological space (X,U,I) is not I-compact. Similar definitions can be given to maximal countably compact space and maximal Lindelof space.We give the characterization of maximal compact space. The other characterizations are proved similarly.

II. A characterization of maximal I-compactness.

2.1 Theorem. Let (X,T,I) be an I-compact ideal topological space. Then (X,T,I) is maximal I-compact iff every I-compact subset of X is closed in X.

Proof. Let (X,T,I) be maximal I-compact. Suppose that there exists an I- compact subset A of X which is not closed. Let T (X − A) be the simple extension **[4]** of T by the set X − A. That is, the members of T (X−A) are subsets of the form

$$U \cup V \cap (X - A)$$

where U and V are members of T. We shall prove that (X, T (X− A), I) is anI-compact which contradicts the maximal I-compactness of (X,T,I).
Let { Gi : i∈ K} an be an open cover of X by members of T (X − A).. Then,

$$Gi = Ui \cup Vi \cap (X - A) \qquad i \in K,$$

where Ui and Vi are members of T for everyi∈ K.
Now,{Ui∪Vi: i∈K}is an T- open cover of X. Since (X,T,I) is I-compact this cover has a a finite subcollection $\{Ui \cup Vi : i \in K_o\}$

where K_o is a finite subset of K,such that$X - \cup\{Ui\cup Vi: i\in K_o\} \in I$.

On the other hand, the collection $\{Ui\cup Vi \cap (X - A):i\in K\}$ which is a T (X − A)-open cover of Ahas a finite subcollection $\{Ui\cup Vi \cap (X - A):i\in K_1\}$,since A is I-compact,where K_1 is a finite subset of K. such that $X- \cup\{Ui\cup Vi \cap (X - A):i\in K_1\}\in I$. Then

$$A - \cup\{Ui\cup Vi \cap (X - A):i\in K_1\}\in I.$$

Also,

$$X - \cup\{Ui\cup Vi\cap (X - A): i\in K_o\} \in I.$$

Now, the union of the above two sets belongs to I containing the set

$$X - \cup\{Ui\cup Vi\cap (X - A): i\in Ko\cup K1\}.$$

Consequently, the later set is a member of I. Thus,(X ,T (X − A), I) is I-compact.

The other of the proof is obvious.

2.2.Corollary. An ideal topological space is maximal I-compact iff every closed subspace of it is maximal I-compact

III. Maximal I-compactness and product.

It is well- known that a compact Hausdorff space is maximal compact.

In **[7]** an example of a maximal non-Hausdorff space is given, and that every maximal compact space is T_1.We give here a simple maximal compact non-Hausdorff space and use it to prove that the product of two maximal non-Hausdorff spaces is not necessarily maximal compact, showing that Tyckonoff theorem is not true for general maximal compact spaces. Since a compact is $\{\varphi\}$- compact, our talk can be drawn into I-compact spaces.

3.1. Example. A non-Hausdorff maximal I-compact space.

Let X be an infinite set and $a\in X$. Topologize X by taking as base singletons of points different from a and subsets of X with finite compliments, containing a. Obviosly X is maximal compact, i.e, maximal $\{\varphi\}$- compact which is not Hausdorff.

3.2 . Example. Product of maximal I-compact spaces is not maximal I-compact.

Let Y be the product space of the space X of Example 3.1 and itself. Then Y is compact, i.e $\{\varphi\}$-compact. It is not maximal $\{\varphi\}$- compact because the topology of Y with base of singletons of points different from (a,a) and subsets of Y containing (a, a) with finite compliments, is a strictly finer topology which is $\{\varphi\}$- compact.

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